— Exercises —

1. Textbook 9.3 (c) p.434

2. **Local vs. global diffeomorphism III.** Show that f is a local diffeomorphism, i.e., $\forall p$ in the domain, $\exists B$ a neighborhood of p such that $f_{|B}$ is a differentiable bijection onto its image, whose inverse is also differentiable. Discuss the injectivity and the surjectivity of f.

$$\begin{aligned} f: \quad \mathbb{R}^3 &\to \mathbb{R}^3 \\ (x,y,z) &\mapsto (e^{2y}+e^{2z}, e^{2x}-e^{2z}, x-y). \end{aligned}$$

— Problems —

3. **Application of the implicit function theorem.** Let

$$\begin{aligned} F: \quad \mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x) \end{aligned}$$

where $\forall j, a_j \in \mathcal{C}^1(\mathbb{R})$. Assume that $\exists x_0, y_0 \in \mathbb{R}$ s.t. the polynomial function

$$P_{x_0} : y \mapsto F(x_0, y)$$

has a simple root at y_0.

Show that there exist V and W, open neighborhoods of x_0 and y_0 respectively, such that $\forall x \in V, \exists! y_x \in W$ s.t. y_x is a simple zero of P_x, and

$$\begin{aligned} V &\to W \\ x &\mapsto y_x \end{aligned}$$

is $\mathcal{C}^1$.

4. **Application of the inverse function theorem.** Denote D be the centered open unit disk in $\mathbb{R}^2$, $\overline{D}$ its closure, and ∂D its boundary. Let $f : \overline{D} \to \mathbb{R}^2$ be a continuous function such that: $f_{|\partial D} = Id_{|\partial D}$; $f_{|D} \in C^1$; $\forall x \in D, J_f(x) \neq 0$.

   (a) Show that if U is open in D, then $f(U)$ is open.

   (b) Show that the image of f is $\overline{D}$.

   (c) Show that $\forall y \in \overline{D}$, $f^{-1}(y)$ is finite.